Itay Neeman: Higher Analogues of PFA

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Lecture 1[1]

We want a forcing axiom that allows us to meet $\aleph_2$ dense sets, and not just $\aleph_1$ as in PFA, and yet has many strong consequences. But there are obstructions (e.g. club guessing at ω_2).

1.1 Poset of finite side conditions of 2 types

Let H be a transitive set (with well-order as part of the structure) with closure properties. Typically this is $H(\kappa)$, but will assume less. Specifically, we assume the following:

1. $\omega, \omega_1 \in H$.
2. If $x, y \in H$, then $x \cap y \in H$.
3. If $f \in H$ is a function, and $x \in H$, then $f(x) \in H$.

We define a collection of nodes $\mathscr{C}$, we specify basic conditions and closure properties of nodes.

1. Each M in $\mathscr{C}$ is of one of the following types:
   (a) (**Type** ω_1) $M \prec_1 H$ (Σ_1 elementary), with $\omega_1 \subseteq M$, $H \models |M| = \omega_1$ (this is the actual ω_1, but usually $\omega_1 = \omega_1^H$).

*Any mistakes in this document can be assumed to be the author's responsibility which were introduced in the editorial process. Please do not read this file.

[1] The notes from this lecture were edited from a transcription by Martin Goldstern.

(b) (**Countable elementary**) $M \prec_1 H$ and $H \models |M| = \omega$.

(c) (**Countable tower**) $|M| \leq \omega$ and is nonempty, $\in$-linear collection of models of type ω_1 in $\mathscr{C}$.

2. (Richness requirement) "closure under $\cap$":

   If $M, N \in \mathscr{C}$ with M of countable elementary type and N of type ω_1 and $N \in M$, then $M \cap N \in \mathscr{C}$. Note that $M \cap N$ is elementary and that N is not required to be countably closed or even club-approachable, but these assumptions would automatically make $M \cap N \in \mathscr{C}$.

   If $M, N \in \mathscr{C}$ and M is a countable tower and N appears in M and $N \cap M \neq \varnothing$, then there is some tower $M' \in \mathscr{C}$ which contains $M \cap N$ and $M' \in N$.

Note 1

If $|M| \leq |N|$ and $M \in N$, then $M \subseteq N$.

1.2 The poset of side conditions

Our side conditions are finite sequences of nodes from $\mathscr{C}$, with additional conditions. If $s = \langle M_i \mid i < n \rangle$ for some $n < \omega$, we say that s is a "two-size" or "two-type" side condition if:

1. $M_i \in \mathscr{C}$ for all i,

2. s is increasing: $M_i \in M_{i+1}$ (note we do not require transitivity here), and

3. closure under $\cap$:

   If $M, N \in s$ with M countable and N of type ω_1, then when M is an elementary type, $M \cap N$ appears as some M_i (and necessarily before M and N); and if M is a tower and $M \cap N$ is not empty, then there is some tower extending $M \cap N$ appears in s.

The conditions 1 and 2 together are called "membership increasing". We will denote the forcing with $\mathbb{P}_{\text{side}}(\mathscr{C})$.

Note 2

We can identify a sequence with the set (order the set in increasing von Neumann rank.)

We define $t \leq s$ if considered as sets $s \subseteq t$. In particular, we are allowed to add new nodes between the existing nodes, not just end extensions.

For $M, N \in s$, we use interval notation, $(M, N]$ etc., to denote the nodes in s which lie in that interval.

Note 3

If s is "membership increasing" and $N, P \in s$, with N appearing before P, then if P has type ω_1, then $N \in P$ and $N \subseteq P$. Proof by downward induction: If N direct predecessor: use note 1. Continue induction.

If P is countable, but N is not, then clearly $N \nsubseteq P$. But if all the nodes in (N, P) are countable, $N \in P$. If N is also countable, then $N \subseteq P$.

We weaken the third condition by only requiring the intersection $M \cap N$ when the interval (M, N) contains only countable nodes. We shall denote this condition as (w3).

Note 4

(w3) implies (3). Proof: again by induction, use note 3. (This is the most important of the small observations!)

Our goal next is to show that $\mathbb{P}_{\text{side}}(\mathscr{C})$ strongly proper. I.e., if s is a condition, $Q \in S$ is an elementary type, then $s \Vdash \dot{G} \cap Q$ is generic for $P \cap Q$.

The proof of this fact is by assigning to each Q a residue function $\text{res}_Q(s)$ such that if $t \in Q$ and $t \leq \text{res}_Q(s)$, then s and t are compatible.

Lecture 2

2.1 The side conditions are strongly proper

Definition 1. Let s be a side condition and $Q \in s$, we define $\text{res}_Q(s) = s \cap Q$.

Lemma 2 (Main Lemma)**.** *1.* $\text{res}_Q(s)$ *is a condition.*

2. If $t \in Q$ *and* $t \leq \text{res}_Q(s)$*, then* t *is compatible with* s*.*

Proof. First assume that Q is of type ω_1. Then $s \cap Q$ is an initial segment of s, and hence a condition. If $t \in Q$ and $t \leq s \cap Q$, then $t \cup s$ is an increasing $\in$-sequence and it satisfies (w3) since given any two nodes in $t \cup s$ with only countable nodes between them, either both are in t or in $s \setminus t$.

If Q is a countable elementary type node, the *residue gaps of s in Q* are the intervals (in s) of the form $[Q \cap N, N)$ for some $N \in Q$ which has type ω_1. Note that a residue gap is disjoint from Q, since any node in $s \cap N$ either lies below $Q \cap N$ or outside of Q.

Take some node $P \in s \setminus Q$ which is below Q. There is some $N \in (P, Q)$ of type ω_1, since otherwise $P \in Q$ by Note 3. Also by Note 3, the largest such N is itself a member of Q. Take the least $N \in Q$ satisfying this, if we show that $Q \cap N$ lies below P in s, then P is in the residue gap. But if $Q \cap N$ lies above P, then as a countable model it would be the case that $P \in Q \cap N$ or there is another $N' \in Q \cap N$ of type ω_1, which is a contradiction to the minimality of N. Therefore P must lie in that residue gap.

Therefore $s \cap Q$ is exactly the part of s below Q minus the residue gaps. From this we can show that $\mathrm{res}_Q(s)$ is indeed a condition. It is $\in$-increasing, since if two nodes are successively in s they are increasing, and in the other case we have some P and N such that P is the maximal node below the residue gap and N is the minimal node above the residue gap, then the gap has the form $[Q \cap N, N)$ and therefore $P \in Q \cap N$ so $P \in N$.

For the closure under the $\cap$, Q is closed under $\cap$ by elementarity and therefore $s \cap Q$ is also closed. We omit the case where Q is not elementary, it is slightly more involved.

Suppose now that $t \in Q$ and $t \leq \mathrm{res}_Q(s)$. Then $s \cup t$ is $\in$-increasing, the only real problem is when verifying the relationship between a node in t and the bottom node of a residue gap $[Q \cap N, N)$, but this is true because N of type ω_1 and so any $P \in t$ below N is a member of N and so of $N \cap Q$.

Finally, we need to verify (w3). If M, W and $W \in M$ are nodes in $t \cup s$ there are four cases to deal with. In the case $M, W \in s$ we get (3) from s, and similarly if $M, W \in t$. The case where $M \in t \setminus s$ and $W \in s \setminus t$ is impossible, since in that case as $t \in Q$ is a finite set, $t \subseteq Q$ and then $W \in Q$, and therefore $W \in \mathrm{res}_Q(s)$ which means that $W \in t$, which is contradictory to the assumption $W \in s \setminus t$.

So we only need to handle the case of $M \in s \setminus t$ and $W \in t \setminus s$. We aim to prove (w3), so we can assume that there are no uncountable nodes in (W, M). If there are residue gaps above W, this implies M belongs to the least one, call it $[N \cap Q, N)$. Moreover there are only ctble nodes in $[N \cap Q, M]$. Let F_W list the countable nodes from $N \cap Q$ up, until (and not including) first node of type ω_1.

Note, lowest node of F_i is $N \cap Q$, and predecessor of W in t belongs to $N \cap Q \cap W$, since it belongs to $t \subseteq Q$, belong to W, and $W \subseteq N$.

If there are no residue gaps above W, reason and define F_W similarly but starting at Q instead of $Q \cap N$.

Let $W_1, \ldots, W_k$ be a list of ω_1 nodes in $t \setminus s$. To close $s \cup t$ under intersections, we need to add for each i the intersections of nodes in F_{W_i} with W_i. Let E_i list these nodes, in order. Add E_i right below W_i. By reasoning above, resulting sequence satisfies (w3). The resulting sequence is still $\in$-increasing: we need only check the borders, upper border is clear since all intersections with W_i belong to W_i, lower border uses last note. $\square$

Lecture 3

We can now prove the following corollaries.

Corollary 3.

1. *If $Q \in s$ is an elementary node, then $s \Vdash \dot{G} \cap Q$ is a generic for $\mathbb{P}_{\mathrm{side}} \cap Q$.*

2. *Moreover, if $Q = Q^* \cap H$, where $Q^* \prec H(\theta)$ for some sufficiently large θ and $\mathbb{P}_{\mathrm{side}} \in Q^*$, then $s \Vdash \dot{G} \cap Q^*$ meets all dense subsets of $\mathbb{P}_{\mathrm{side}}$ in Q^*.*

3. *Additionally, any $t \in Q^*$ has an extension t' with $Q \in t'$.*

Proof. Use the Main Lemma for (1). The second item follows from the first, since for every dense $D \in Q^*$, $D \cap Q = D \cap Q^*$ is dense in $\mathbb{P}_{\text{side}} \cap Q = \mathbb{P}_{\text{side}} \cap Q^*$. For the third part, use the Main Lemma with $s = \{Q\}$. $\square$

Corollary 4. *If $\{N \in \mathscr{C} \mid N$ has type $\omega_1\}$ is stationary in $\mathcal{P}_{\omega_2}(H)$,[1] then $\mathbb{P}_{\text{side}}$ preserves ω_2. Similarly, if $\{M \in \mathscr{C} \mid M$ is a countable elementary node$\}$ is stationary, ω_1 is preserved. Moreover, $|H|$ will be collapsed to ω_2.*

Note that the generic filter will be a collection of nodes, and while it is not linearly ordered, the collection of nodes of type ω_1 will be a linearly ordered and will have order type ω_2.

All this is fine, but now we need to find suitable collections of nodes $\mathscr{C}$ in order to ensure that things work out nicely. Fix some θ such that $\operatorname{cf}(\theta) \geq \omega_2$, and fix some $f\colon H(\theta)^{<\omega} \to H(\theta)$. We define $\mathscr{C}(\theta, f)$ to include all:

- $N \prec_1 H(\theta)$ which is internal on a club, closed under f and has size $\aleph_1$.
- $M \prec_1 H(\theta)$ which is closed under f and has size $\aleph_0$.
- All the tower nodes closed under intersections (i.e. M is a tower and $N \in M$, then $M \cap N \in N$).

Typically f is some Skolem function that codes "everything we need it to code".

3.1 Baby properness

Definition 5. We say that a forcing $\mathbb{P}$ is *baby $\{\omega, \omega_1\}$-proper* if there is some θ and a function f and a function mc defined on $\mathscr{C}(\theta, f)$ such that:

1. $\operatorname{mc}(M)$ is an open subset of $\mathbb{P}$.
2. If M is a countable elementary type, every $p \in \operatorname{mc}(M)$ is a master condition for M (i.e. $p \Vdash \dot{G} \cap M$ meets every dense $D \in M$).
3. $\operatorname{mc}(M \cap N) \supseteq \operatorname{mc}(M) \cap \operatorname{mc}(N)$ when $N \in M$ and N has type ω_1 with $M \cap N \neq \varnothing$).
4. If M is a tower, then for every $N \in M$, $\operatorname{mc}(N)$ is dense in $\operatorname{mc}(M)$.
5. If $M_1 \subseteq M_2$ are two towers, then $\operatorname{mc}(M_1) \supseteq \operatorname{mc}(M_2)$.
6. For every $s \in \mathbb{P}_{\text{side}}(\mathscr{C}(\theta, f))$, every elementary type node $Q \in s$, every condition $p \in \bigcap\{\operatorname{mc}(M) \mid m \in \operatorname{res}_Q(s)\}$ extends to a $p' \in \bigcap\{\operatorname{mc}(M) \mid M \in s\}$.

The conditions (1)–(5) simply say that $\operatorname{mc}(M)$ is a distinguished set of master conditions. The sixth condition is in fact the main ingredient of the definition.

Our forcing axiom is the assertion that if $\mathbb{P}$ is baby $\{\omega, \omega_1\}$-proper, then for every collection $\mathcal{D}$ of $\aleph_2$ dense sets there is a $\mathcal{D}$-generic filter.

[1]Here a stationary is defined via algebras.

Note 5

If $s = \{M_1, \ldots, M_k\}$ has only elementary nodes of size $\aleph_0$, or only nodes of type ω_1, then (6) is reduced to

(∗) For every elementary Q, every $p \in Q$ extends to $p' \in \mathrm{mc}(Q)$.

The proof is by inductively applying (∗) and using elementarity.

If s mixes the types of elementary models, then if Q is countable, $N \in Q$ is of type ω_1, then there is a residue gap, where we cannot use the inductive proof. Moreover, if s has a tower M above an uncountable Q, then (6) tells us that $p \in Q$ extends to a master condition over all the nodes in the tower M.

Note 6

The last fact mentioned in the previous note rules out posets which kill club guessing sequences, and therefore allow us to even have a chance of defining a forcing axiom which will allow the intersection of $\aleph_2$ dense subsets.

Note 7

$\mathbb{P}_{\mathrm{side}}$ satisfies (6) is exactly our Main Lemma by taking $\mathrm{mc}(M) = \{s \in \mathbb{P}_{\mathrm{side}} \mid M \in s\}$.

Note 8

Every ccc forcing is baby $\{\omega, \omega_1\}$-proper since we can take $\mathrm{mc}(M) = \mathbb{P}$. Collapsing cardinals to ω_2 can also be done by such forcing (e.g. using $\mathbb{P}_{\mathrm{side}}$). The class is closed under compositions. So it seems like a very rich class of forcings. Analogously at $\aleph_1$, many consequences of PFA obtained using compositions of collapse to $\aleph_1$ and applying a ccc forcing. But at $\aleph_2$ unfortunately these compositions seem less powerful; it is harder to find powerful ccc posets at $\aleph_2$.

We can iterate baby $\{\omega, \omega_1\}$-proper posets, using finite supports iteration with three-size side conditions. Starting from a supercompact, using a Laver function in the standard way, this leads to the consistency of the following forcing axiom: For every baby $\{\omega, \omega_1\}$-proper poset $\mathbb{P}$, and every collection $\mathcal{F}$ of $\aleph_2$ dense subsets of $\mathbb{P}$, there is a filter over $\mathbb{P}$ meeting all dense sets in $\mathcal{F}$.

Note 10

The forcing axiom implies $\square_{\omega_1,\omega}$. In fact, the forcing axiom implies $\square_{\omega,\omega_1}$ with tail-end agreement, which in turn implies the existence of an $\aleph_2$-Aronszajn tree with an ascent path, hence a non-special $\aleph_2$-Aronszajn tree. You can read more on this in [1].

Note 11

There are strengthening of the forcing axioms, relaxing condition (6) in the definition of baby $\{\omega, \omega_1\}$-properness, which imply $\neg\Box_\kappa$ for all $\kappa \geq \omega_2$. There are also other strengthenings which give an analogue of a consequence of MRP and imply $2^{\aleph_0} = \aleph_3$.

Post-Lecture

The following is material supplied by Itay which was not covered in the lectures due to time constraints.

We can relax condition (6) in the definition of baby $\{\omega, \omega_1\}$-proper by restricting it only to the following cases:

(a) $\operatorname{res}_Q(s) = \varnothing$, or

(b) Q is countable elementary, and there are no countable elementary nodes in $\operatorname{res}_Q(s)$.

Then we require that $p \in \bigcap\{\operatorname{mc}(M) \mid M \in \operatorname{res}_Q(s)\}$ extends to $p' \in \bigcap\{\operatorname{mc}(M) \mid M \in s\}$. We call this (relaxed 6).

Case (a) for Q of type ω_1 very close to $(*)$, *but* a bit stronger because of tower nodes. For example if there is a tower $M \in s$ above Q, then (relaxed 6) leads to a master condition p' for all (countably many) $N \in M$. This extra strength is needed to avoid killing club guessing, as mentioned in Note 6.

In case (b), say that a tower U *subsumes* $\operatorname{res}_Q(s)$ if every type ω_1 node $N \in \operatorname{res}_Q(s)$ belongs to U, and every tower node $M \in \operatorname{res}_Q(s)$ is contained in U.

We can relax (6) even further by weakening case (b) and only require existence of p' if there is a tower node U so that U subsumes $\operatorname{res}_Q(s)$ and $p \in \operatorname{mc}(U)$. We call this (very relaxed 6).

Definition 6. A forcing is *very relaxed* $\{\omega, \omega_1\}$*-proper* if there are θ, f, and mc satisfying conditions (1)-(5) and (very relaxed 6).

We can get a forcing axiom for this class, with the standard strengthening of maintaining stationarity: For every very relaxed $\{\omega, \omega_1\}$-proper forcing $\mathbb{P}$, every collection $\mathcal{F}$ of $\aleph_2$ dense subsets of $\mathbb{P}$, and every $\mathbb{P}$-name $\dot{S}$ for a stationary subset of ω_2 with points of cofinality ω_1, there is a filter G meeting all sets in $\mathcal{F}$ and so that $\dot{S}^G$ is stationary.

Note 12

This axiom implies $\neg\Box_\kappa$ for all $\kappa \geq \omega_2$. Proved using an elaboration on $\mathbb{P}_{\text{side}}$ that adds club which "anti-threads" a given $\Box_\kappa$ sequence on a stationary set.

Note 13

Using some hybrid of the baby axiom and very relaxed axiom, can get an analogue/variant of MRP, which implies there is a well-ordering of $H(\aleph_3)$ of order type ω_3 definable from parameters over $H(\aleph_3)$. In particular then $2^{\aleph_0} = \aleph_3$.

Note 14

Many questions are left open. The exact differences between the axiom variants not entirely clear (for example does the baby version negate $\square_\kappa$ for $\kappa \geq \omega_2$?). The main open question is to find the right axiom. We need to find out how to control what happens inside residue gaps of s in Q. Condition (6) allows any behavior in the residue gaps, relaxed and very relaxed versions limit possibly behaviors. Other variants of (6) may be needed with better limits on the allowed behavior in the residue gaps.

Bibliography

[1] Itay Neeman, *Two applications of finite side conditions at ω_2*, Arch. Math. Logic **56** (2017), no. 7-8, 983–1036. MR 3696074